A CHARACTERIZATION OF THE EVERYWHERE REGULAR SOLUTION OF THE REDUCED WAVE EQUATION

BY

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Introduction. The investigation herein is concerned with characterizing a class of everywhere regular solutions of the reduced wave equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + u = 0, \qquad u = u(x, y) = u(r\cos\theta, r\sin\theta).$$

It is known [1] that any solution regular for $r > r_0 > 0$ can be uniquely decomposed into the sum of two solutions, U and V, possessing the following properties. U is an everywhere regular solution. V is a regular solution for $r > r_0$ and it satisfies the Sommerfeld radiation condition

$$\frac{\partial V}{\partial r} + iV = O(r^{-3/2})$$

for large r.

Furthermore, at infinity, V necessarily has the following form [1]

$$V = \left(\frac{\pi}{2} r\right)^{-1/2} e^{-i(r-\pi/4)} f(\theta) + O(r^{-3/2}),$$

where $f(\theta) = \sum \beta_n e^{in\theta}$ is an entire analytic function. Conversely, if $f(\theta)$ is given, then $V = \sum i^{-n}\beta_n H_n^{(2)}(r)e^{in\theta}$ ($H_n^{(2)}(r)$ is a Hankel function). Lastly, at infinity U necessarily has the form

$$U = (2\pi r)^{-1/2}\left[g(\theta)e^{i(r-\pi/4)} + g(\theta+\pi)e^{-i(r-\pi/4)}\right] + O(r^{-3/2}),$$

where $g(\theta) \equiv \sum \alpha_n e^{in\theta}$ is sufficiently regular, say a function of bounded variation. Conversely, if $g(\theta)$ is given, then

$$U = \sum i^n \alpha_n J_n(r) e^{in\theta}.$$

The purpose of this paper is to consider a different characterization of the everywhere regular solution. Specifically, U will now be determined from $F(\theta) \equiv \sum i^n \alpha_n e^{in\theta}$; that is, from $\int_0^\infty U dr$, since $\int_0^\infty J_n(r) dr = 1$ $(n = 0, 1, 2, \cdots)$. This characterization is such that the corresponding solution U has an explicit integral representation.

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The principal difficulty involved in the justification of these assertions is related to the fact, (1.15), that the integral $\int_0^\infty |U| dr$ does not converge uniformly; this property also follows from a theorem of F. Rellich [2] which asserts that U is at least of order $r^{-1/2}$ for large r. Nevertheless, it will be established that the integral $\int_0^\infty U dr$ does indeed converge uniformly provided that $F(\theta)$ is a sufficiently regular periodic function. A precise statement of all the results alluded to is given in the following:

THEOREM. *Assume that $u(x, y)$ is of class C^2 and everywhere satisfies the reduced wave equation. Furthermore, assume that uniformly in θ the integral*

$$\int_0^\infty u dr = F(\theta), \qquad u = u(x, y) = u(r\cos\theta, r\sin\theta),$$

$F(\theta)$ being of period 2π and fulfilling a uniform Hölder condition with an exponent $\alpha > 1/2$. Then the integral value problem has a unique solution and the solution has the integral representation

$$u(x, y) = \frac{1}{2\pi}\int_0^{2\pi} F(\phi)\cos(r\sin(\phi-\theta))d\phi$$

$$+ \frac{1}{2\pi^2}\int_0^{2\pi} d\phi F(\phi)\int_0^{\pi/2}\frac{[\sin(r\sin(\tau+\phi-\theta)) + \sin(r\sin(\tau-\phi+\theta))]d\tau}{\sin\tau}.$$

1. The uniqueness theorem. The details herewith used for the establishment of the uniqueness theorem will provide heuristic motivation for the beginning development of the existence theorem. The uniqueness theorem itself, (1.1), will be an easy consequence of

THEOREM 1. *Assume that $u(x, y)$ is of class C^2 and everywhere satisfies the reduced wave equation*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + u = 0, \qquad u = u(x, y) = u(r\cos\theta, r\sin\theta). \tag{1.1}$$

Furthermore, assume that uniformly in θ the integral

$$\int_0^\infty u dr = F(\theta), \qquad (F(\theta + 2\pi) = F(\theta)). \tag{1.2}$$

Then the solution $u(x, y)$ has the series representation

$$u(x,y)y = \sum_{k=0}^\infty J_k(r)[a_k\cos k\theta + b_k\sin k\theta], \tag{1.3}$$

$J_k(r)$ denoting the Bessel function of the first kind of order k, and

$$a_k + ib_k = \frac{1}{\pi}\int_0^{2\pi} F(\phi)e^{ik\phi}d\phi, \quad a_0 = \frac{1}{2\pi}\int_0^{2\pi} F(\phi)d\phi \quad (k = 1, 2, \cdots). \tag{1.4}$$

Proof. Since $u(x, y)$ is of class C^2,

$$u(x, y) = \sum_{k=0}^{\infty} [A_k(r) \cos k\theta + B_k(r) \sin k\theta], \tag{1.5}$$

where

$$A_k + iB_k = \frac{1}{\pi}\int_0^{2\pi} u(r, \phi)e^{ik\phi}d\phi, \; A_0 = \frac{1}{2\pi}\int_0^{2\pi} u(r, \phi)d\phi \; (k = 1, 2, \cdots). \tag{1.6}$$

Hence, because of (1.1),

$$\frac{d^2A_k}{dr^2} + \frac{1}{r}\frac{dA_r}{dr} + A_r = -\frac{1}{\pi r^2}\int_0^{2\pi}\frac{\partial^2 u}{\partial\phi^2}\cos k\phi d\phi = -\frac{k^2}{r^2}A_k, \tag{1.7}$$

which implies

$$A_k = a_kJ_k(r), \qquad B_k = b_kJ_k(r) \qquad (k = 0, 1, 2, \cdots), \tag{1.8}$$

a_k and b_k being numerical constants. Substituting (1.8) into (1.6) gives

$$(a_k + ib_k)J_k(r) = \frac{1}{\pi}\int_0^{2\pi} u(r, \phi)e^{ik\phi}d\phi, \quad a_0J_0(r) = \frac{1}{2\pi}\int_0^{2\pi} u(r, \phi)d\phi. \tag{1.9}$$

Finally, because of the uniform convergence of the integral (1.2), the known fact [3]

$$\int_0^{\infty} J_k(n)dr = 1 \qquad (k = 0, 1, 2, \cdots), \tag{1.10}$$

and the equation (1.9), the relations (1.4) are valid. Q.E.D.

The following three corollaries are readily deducible.

(1.11) Corollary 1 (Uniqueness Theorem). *If $u(x,y)$ satisfies the hypotheses of Theorem 1 and if $F(\theta) \equiv 0$, then $u(x, y) \equiv 0$.*

(1.12) Corollary 2 (Representation Theorem). *If $u(x, y)$ satisfies the hypotheses of Theorem 1 and if the order of summation and integration in (1.3) is interchangeable, then $u(x, y)$ has the integral representation*

$$u(x, y) = \frac{1}{\pi}\int_0^{2\pi} F(\phi)E(r, \phi - \theta)d\phi, \tag{1.13}$$

where

$$E(r, \theta) \equiv \frac{1}{2}J_0(r) + \sum_{k=1}^{\infty} J_k(r) \cos k\theta. \tag{1.14}$$

(1.15) Corollary 3 (Convergence Theorem). *If $u(x, y)$ is an everywhere regular solution of the reduced wave equation and if the integral $\int_0^\infty |u| dr$ is uniformly convergent, then $u \equiv 0$.*

Proof. Using (1.9) it follows that

$$| a_k + ib_k | \int_0^R | J_k(r) | \, dr \leqq \int_0^{2\pi} d\theta \int_0^R | u | \, dr \leqq \int_0^{2\pi} d\theta \int_0^\infty | u | \, dr < \infty$$

$$(R > 0).$$

Now, since

$$\lim_{R \to \infty} \int_0^R | J_k(r) | \, dr = \infty \qquad (k = 0, 1, 2, \cdots),$$

it necessarily follows that $a_k = b_k = 0$ $(k = 0, 1, 2, \cdots)$. That is, $u \equiv 0$.

2. **Summation of** $E(r, \theta)$. It will be shown here that the kernel $E(r, \theta)$ can be explicitly summed and in fact has the representation

$$\text{(2.1)} \quad \begin{aligned} E(r, \theta) &= \frac{1}{2} \cos (r \sin \theta) \\ &+ \frac{1}{2\pi} \int_0^{\pi/2} \frac{[\sin (r \sin (\tau + \theta)) + \sin (r \sin (\sin (\tau - \theta))]}{\sin \tau} \, d\tau. \end{aligned}$$

This relation obviously will be established on the basis of the next two lemmas.

(2.2) LEMMA 1. *The series*

$$\frac{1}{2} J_0(r) + \sum_{k=1}^{\infty} J_k(r) \cos 2k\theta = \frac{1}{2} \cos (r \sin \theta).$$

Proof. The result is well-known, see [4].

(2.3) LEMMA 2. *The series*

$$\sum_{k=0}^{\infty} J_{2k+1}(r) \cos (2k + 1)\theta = \frac{1}{2\pi} \int_0^{\pi/2} \frac{[\sin (r \sin (\tau + \theta)) + \sin (r \sin (\tau - \theta))]}{\sin \tau} \, d\tau.$$

REMARK. By means of a rather long and delicate limit process, the author showed that the series could be transformed into the integral form. Conversely, in an exceedingly simple manner, the referee showed that the integral could be transformed into the series form. The proof of the referee will be given forthwith, but not before expressing my sincere appreciation for his helpful comments.

Proof. Let $S(\theta)$ denote the integral expression. Then $S(\theta)$ possesses the following two properties:

$$S(-\theta) = S(\theta), \qquad S(\theta + \pi) = - S(\theta).$$

Thus, the Fourier series for $S(\theta)$ contains only terms of the form $\cos (2k+1)\theta$.

Hence the integral is identical with the series if and only if their corresponding Fourier coefficients are equal. That is, if

$$J_{2k+1}(r) = S_k$$
$$= \frac{1}{2\pi^2}\int_0^{2\pi} d\theta \cos(2k+1)\theta \int_0^{\pi/2} \frac{[\sin(r\sin(\tau+\theta)) + \sin(r\sin(\tau-\theta))]}{\sin\tau}\, d\tau.$$

If the order of integration is interchanged, then one finds that

$$S_k = \frac{1}{\pi^2}\int_0^{\pi/2} d\tau \frac{\sin(2k+1)\tau}{\sin\tau}\int_0^{2\pi} \sin(r\sin\theta)\sin(2k+1)\theta d\theta$$
$$= \frac{2}{\pi}\int_0^{\pi/2} \frac{\sin(2k+1)\tau}{\sin\tau} J_{2k+1}(r) dr$$
$$= \frac{2}{\pi} J_{2k+1}(r)\int_0^{\pi/2}\left[1 + 2\sum_{n=1}^{k}\cos(2n\tau)\right] d\tau = J_{2k+1}(r),$$

which completes the proof of the lemma.

Because of (2.1), the formula (1.13) can be expressed in the following form:

$$u(x, y) = u(r, \theta; F) = \frac{1}{2\pi}\int_0^{2\pi} F(\phi)\cos(r\sin(\phi-\theta))d\phi$$
(2.4)
$$+ \frac{1}{2\pi^2}\int_0^{2\pi} d\phi F(\phi)\int_0^{\pi/2}\frac{[\sin(r\sin(\tau+\phi-\theta)) + \sin(r\sin(\tau-\phi+\theta))]}{\sin\tau}\, d\tau.$$

The equation (2.4) can be put into a more convenient form if we notice that any $F(\theta)$ of period 2π can be decomposed into the sum of two functions:

(2.5) $$F_1(\theta) \equiv \frac{F(\theta) + F(\theta+\pi)}{2}, \qquad F_2(\theta) \equiv \frac{F(\theta) - F(\theta+\pi)}{2},$$

where

(2.6) $$F_1(\theta+\pi) = F_1(\theta), \qquad F_2(\theta+\pi) = -F_2(\theta).$$

Hence, since

(2.7) $$u_1 \equiv u(r, \theta; F_1) = \frac{1}{\pi}\int_0^{\pi} F_1(\phi)\cos(r\sin(\phi-\theta))d\phi$$

and

$$u_2 \equiv u(r, \theta; F_2)$$
(2.8)
$$= \frac{1}{\pi^2}\int_0^{\pi} d\phi F_2(\phi)\int_0^{\pi/2}\frac{[\sin(r\sin(\tau+\phi-\theta)) + \sin(r\sin(\tau-\phi+\theta))]}{\sin\tau}\, d\tau,$$

it follows that

(2.9) $$u = u(r, \theta; F) = u(r, \theta; F_1 + F_2) = u_1 + u_2.$$

This representation for u will be especially useful when verifying the integral condition

(2.10) $$\int_0^\infty u dr = F(\theta).$$

3. **Evaluation of $\int_0^\infty u_1 dr$.** The principal goal of the remainder of the paper is to show that the u defined by (2.4) fulfills the integral condition (1.2), that is, (2.10), if $F(\theta)$ satisfies a Hölder condition with exponent $\alpha > 1/2$. This investigation will now be commenced by proving the next

(3.1) LEMMA 3.

$$\int_0^\infty u(r, \theta; 1) dr = 1.$$

Proof. Because of (2.7)

(3.2) $$u(r, \theta; 1) = \frac{2}{\pi} \int_0^{\pi/2} \cos (r \sin \tau) d\tau.$$

Thus,

$$\int_0^\infty u(r, \theta; 1) dr = \lim_{N\to\infty} \frac{2}{\pi} \int_0^{\pi/2} \frac{\sin (N \sin \tau)}{\sin \tau} d\tau$$

$$= \lim_{N\to\infty} \frac{2}{\pi} \int_0^1 \frac{\sin Nx}{x(1 - x^2)^{1/2}} dx$$

(3.3) $$= \lim_{N\to\infty} \frac{2}{\pi} \int_0^1 \frac{\sin Nx}{x} dx + \lim_{N\to\infty} \frac{2}{\pi} \int_0^1 \frac{1}{x} \left[\frac{1}{(1 - x^2)^{1/2}} - 1 \right] \sin Nx dx$$

$$= 1 + \lim_{N\to\infty} \frac{2}{\pi} \int_0^1 \frac{1}{x} \left[\frac{1}{(1 - x^2)^{1/2}} - 1 \right] \sin Nx dx.$$

Now, because of the Riemann-Lebesgue theorem = RLT, the last limit is zero. Hence, the lemma is proved.

The remainder of this section will be concerned with proving

(3.4) LEMMA 4. *If $F_1(\theta) \in H^{\alpha>0}$ (the class of functions satisfying a uniform Hölder condition with exponent $\alpha > 0$), then uniformly in θ the integral*

$$\int_0^\infty u(r, \theta; F_1) dr = F_1(\theta).$$

Proof. Because of Lemma 3,

$$\int_0^\infty u_1 dr - F_1(\theta)$$

$$(3.5)\qquad = \lim_{N\to\infty} \frac{1}{\pi}\int_0^N dr \int_0^\pi [F_1(\phi) - F_1(\theta)] \cos (r \sin (\phi - \theta)) d\phi$$

$$= \lim_{N\to\infty} \frac{1}{\pi}\int_0^\pi \frac{[F_1(\phi) - F_1(\theta)]}{\sin (\phi - \theta)} \sin (N \sin (\phi - \theta)) d\phi \equiv \lim_{N\to\infty} I(N).$$

Furthermore,

$$\lim_{N\to\infty} I(N) = \lim_{N\to\infty} \frac{1}{\pi}\int_0^\pi \frac{[F_1(\tau + \theta) - F_1(\theta)]}{\sin \tau} \sin (N \sin \tau) d\tau$$

$$= \lim_{N\to\infty} \frac{1}{\pi}\int_0^{\pi/2} \frac{[F_1(\tau + \theta) - F_1(\theta)]}{\sin \tau} \sin (N \sin \tau) d\tau$$

$$(3.6)\qquad + \lim_{N\to\infty} \frac{1}{\pi}\int_0^{\pi/2} \frac{[F_1(\tau + \theta + \pi/2) - F_1(\theta)]}{\cos \tau} \sin (N \cos \tau) d\tau$$

$$= \lim_{N\to\infty} I_1(N) + \lim_{N\to\infty} I_2(N).$$

Setting $x = \sin \tau$ in $I_1(N)$ and passing to the limit yields

$$(3.7)\qquad \lim_{N\to\infty} I_1(N) = \lim_{N\to\infty} \frac{1}{\pi}\int_0^1 \frac{[F_1(\tau + \theta) - F_1(\theta)]}{x(1 - x^2)^{1/2}} \sin (Nx) dx = 0,$$

the RLT and the fact that $F_1(\theta) \in H^{\alpha>0}$ being taken into account. In a similar fashion the subsequent equation is derived.

$$(3.8)\qquad \lim_{N\to\infty} I_2(N) = \lim_{N\to\infty} \frac{1}{\pi}\int_0^1 \frac{[F_1(\tau + \theta + \pi/2) - F_1(\theta)]}{x(1 - x^2)^{1/2}} dx = 0.$$

Furthermore, the convergence to zero is uniform in θ in both of the relations (3.7) and (3.8). Finally, because of (3.5), (3.7) and (3.8), the proof of Lemma 4 is completed.

4. **Integral representation for $\int_0^\infty u_2 dr$.** The goal of this section is to prove the following

(4.1) LEMMA 5. *If $F_2(\theta) \in H^{\alpha>0}$ and if*

$$(4.2)\qquad \begin{aligned} \Omega(\phi, \tau, \theta) \equiv{} & F_2(\phi + \theta - \tau) + F_2(-\phi + \theta + \tau) - F_2(-\phi + \theta - \tau) \\ & - F_2(\phi + \theta + \tau), \end{aligned}$$

then uniformly in θ the integral

$$(4.3)\qquad \int_0^\infty u_2 dr = \frac{1}{\pi^2}\int_0^{\pi/2} \frac{d\tau}{\sin \tau}\int_0^{\pi/2} \frac{\Omega(\phi, \tau, \theta)}{\sin \phi} d\phi \equiv \Lambda[F_2].$$

Proof. The definition (2.8) states that

$$u_2(r, \theta; F_2)$$

$$(4.4)\quad = \frac{1}{\pi^2}\int_0^\pi d\phi F_2(\phi)\int_0^{\pi/2}\frac{[\sin(r\sin(\tau+\phi-\theta))+\sin(r\sin(\tau-\phi+\theta))]}{\sin\tau}d\tau.$$

Using the relation $F_2(\phi+\pi)=-F_2(\phi)$ and the transformation $\bar{\phi}=\phi-\theta$, the relation (4.4) becomes

$$(4.5)\quad u_2 = \frac{1}{\pi^2}\int_0^\pi d\phi F_2(\phi+\theta)\int_0^{\pi/2}\frac{[\sin(r\sin(\tau+\phi))+\sin(r\sin(\tau-\phi))]}{\sin\tau}d\tau,$$

the bar over $\bar{\phi}$ having been deleted. By interchanging the order of integration in (4.5), one gets

$$u_2 = \frac{1}{\pi^2}\int_0^{\pi/2}\frac{d\tau}{\sin\tau}\int_0^\pi F_2(\phi+\theta)[\sin(r\sin(\tau+\phi))+\sin(r\sin(\tau-\phi))]d\phi$$

$$(4.6)\quad = \frac{1}{\pi^2}\int_0^{\pi/2}\frac{d\tau}{\sin\tau}\int_\tau^{\pi+\tau}[F_2(\phi+\theta-\tau)+F_2(-\phi+\theta+\tau)]\sin(r\sin\phi)d\phi$$

$$= \frac{1}{\pi^2}\int_0^{\pi/2}\frac{d\tau}{\sin\tau}\int_0^\pi[F_2(\phi+\theta-\tau)+F_2(-\phi+\theta+\tau)]\sin(r\sin\phi)d\phi.$$

Thus,

$$(4.7)\quad \int_0^N u_2 dr = \frac{1}{\pi^2}\int_0^{\pi/2}\frac{d\tau}{\sin\tau}\int_0^\pi[F_2(\phi+\theta-\tau)+F_2(-\phi+\theta+\tau)] \cdot\left[\frac{1-\cos(N\sin\phi)}{\sin\phi}\right]d\phi,$$

and (4.7) can be put in the form

$$(4.8)\quad \begin{aligned}\int_0^N u_2 dr = {} & \frac{1}{\pi^2}\int_0^{\pi/2}\frac{d\tau}{\sin\tau}\int_0^{\pi/2}[F_2(\phi+\theta-\tau)+F_2(-\phi+\theta+\tau)] \cdot\left[\frac{1-\cos(N\sin\phi)}{\sin\phi}\right]d\phi \\ & + \frac{1}{\pi^2}\int_0^{\pi/2}\frac{d\tau}{\sin\tau}\int_{\pi/2}^{\pi}[F_2(\phi+\theta-\tau)+F_2(-\phi+\theta+\tau)] \cdot\left[\frac{1-\cos(N\sin\phi)}{\sin\phi}\right]d\phi.\end{aligned}$$

Herewith, setting $\phi=\arc\sin x\equiv\sin^{-1}x$ in the first integral and $\pi-\sin^{-1}x$ in the second integral of the right member of the last equation, (4.8) becomes

$$(4.9)\quad \begin{aligned} \int_0^N u_2 dr = & \frac{1}{\pi^2}\int_0^{\pi/2}\frac{d\tau}{\sin\tau}\int_0^1 [F_2(\sin^{-1}x+\theta-\tau)+F_2(-\sin^{-1}x+\theta+\tau)] \\ & \cdot\left[\frac{1-\cos Nx}{x(1-x^2)^{1/2}}\right]dx \\ & -\frac{1}{\pi^2}\int_0^{\pi/2}\frac{d\tau}{\sin\tau}\int_0^1 [F_2(-\sin^{-1}x+\theta-\tau)+F_2(\sin^{-1}x+\theta+\tau)] \\ & \cdot\left[\frac{1-\cos Nx}{x(1-x^2)^{1/2}}\right]dx. \end{aligned}$$

Now, because of (4.2), the equation (4.9) is the same as

$$(4.10)\quad \int_0^N u_2 dr = \frac{1}{\pi^2}\int_0^{\pi/2}\frac{d\tau}{\sin\tau}\int_0^1 \Omega(\sin^{-1}x,\tau,\theta)\left[\frac{1-\cos(Nx)}{x(1-x^2)^{1/2}}\right]dx.$$

In order to determine the limiting form of (4.10) it is sufficient to invoke Lebesgue's theorem concerning the passage to the limit under the integral sign. That Lebesgue's theorem is applicable is clear since Ω satisfies a Hölder inequality (4.13). As $F_2 \in H^{\alpha>0}$,

$$(4.11)\quad |\Omega(\phi,\tau,\theta)| \leqq 2A(2\tau)^\alpha, \qquad |\Omega(\phi,\tau,\theta)| \leqq 2A(2\phi)^\alpha,$$

where A is the Hölder constant associated with $F_2(\theta)$. So,

$$(4.12)\quad |\Omega(\phi,\tau,\theta)| \leqq 2^{1+\alpha}A(\tau\phi)^{\alpha/2},$$

or

$$(4.13)\quad |\Omega(\sin^{-1}x,\tau,\theta)| \leqq 2^{1+\alpha}A(\tau\sin^{-1}x)^{\alpha/2}.$$

The relation (4.10) now becomes

$$(4.14)\quad \begin{aligned} \lim_{N\to\infty}\int_0^N u_2 dr = & \frac{1}{\pi^2}\int_0^{\pi/2}\frac{1}{\sin\tau} \\ & \cdot\left\{\lim_{N\to\infty}\int_0^1 \Omega(\sin^{-1}x,\tau,\theta)\left[\frac{1-\cos(Nx)}{x(1-x^2)^{1/2}}\right]dx\right\}d\tau. \end{aligned}$$

Moreover, because of (4.12) and the RLT, (4.14) becomes

$$(4.15)\quad \begin{aligned} \lim_{N\to\infty}\int_0^N u_2 dr & = \frac{1}{\pi^2}\int_0^{\pi/2}\frac{d\tau}{\sin\tau}\int_0^1\frac{\Omega(\sin^{-1},x,\tau,\theta)}{x(1-x^2)^{1/2}}dx \\ & = \frac{1}{\pi^2}\int_0^{\pi/2}\frac{d\tau}{\sin\tau}\int_0^{\pi/2}\frac{\Omega(\phi,\tau,\theta)}{\sin\phi}d\phi \equiv \Lambda[F_2]. \end{aligned}$$

Finally, by reason of (4.12), the convergence of the integral (4.15) is uniform in θ. That is, given an $\epsilon>0$, there exists an integer $N_0(\epsilon)$ so that

$$\left| \int_0^N u_2 dr - \Lambda[F_2] \right| < \epsilon,$$

whenever $N \geqq N_0(\epsilon)$. Q.E.D.

5. **Verification that** $\Lambda[F_2] = F_2(\theta)$. This is true because of the next lemma:

(5.1) LEMMA 6. *If* $F_2(\theta) \in H^{\alpha > 1/2}$, *then* $\Lambda[F_2] = F_2(\theta)$.

Proof. A theorem of S. Bernstein [5] and the hypotheses of the lemma imply that

$$F_2(\theta) = \sum_{k=0}^{\infty} [a_{2k+1} \cos (2k+1)\theta + b_{2k+1} \sin (2k+1)\theta], \tag{5.2}$$

the series converging uniformly to $F_2(\theta)$ and

$$\sum_{k=0}^{\infty} [|a_{2k+1}| + |b_{2k+1}|] < \infty. \tag{5.3}$$

Moreover, it follows from (4.2) and (5.2) that

$$\begin{aligned} \Omega(F_2) &= 4 \sum_{k=0}^{\infty} a_{2k+1} \sin (2k+1)\phi \sin (2k+1)\tau \cos (2k+1)\theta \\ &\quad + 4 \sum_{k=0}^{\infty} b_{2k+1} \sin (2k+1)\phi \sin (2k+1)\tau \sin (2k+1)\theta. \end{aligned} \tag{5.4}$$

Consequently, as the series (5.4) is uniformly, absolutely convergent,

$$\begin{aligned} \int_\epsilon^\pi \frac{\Omega}{\sin \phi} d\phi &= 4 \sum_{k=0}^{\infty} [a_{2k+1} \cos (2k+1)\theta \\ &\quad + b_{2k+1} \sin (2k+1)\theta] \sin (2k+1)\tau \int_\epsilon^\pi \frac{\sin (2k+1)\phi}{\sin \phi} d\phi, \end{aligned} \tag{5.5}$$

with an arbitrary $\epsilon > 0$. An auxiliary inequality is now needed before the limiting form of (5.5) can be computed. The desired inequality is the following:

$$\left| \int_\epsilon^\pi \frac{\sin (2k+1)\phi}{\sin \phi} d\phi \right| \leqq \pi + 1 \qquad (k = 0, 1, 2, \cdots). \tag{5.6}$$

Proof. Since

$$\frac{\sin (2k+1)\phi}{\sin \phi} = 1 + 2 \sum_{n=1}^{k} \cos 2n\phi, \tag{5.7}$$

it follows that

$$(5.8)\qquad \int_{\epsilon}^{\pi/2} \frac{\sin (2k+1)\phi}{\sin \phi}\, d\phi = \frac{\pi}{2} - \epsilon - \sum_{n=1}^{k} \frac{\sin 2n\epsilon}{n}.$$

Consequently, as

$$(5.9)\qquad \left| \sum_{n=1}^{k} \frac{\sin nx}{n} \right| \leqq \frac{\pi}{2} + 1 \qquad (k = 1, 2, \cdots),$$

the relation (5.6) is implied by (5.8) and (5.9). Q.E.D.

The inequalities (5.3) and (5.6) enable us to conclude that, as ϵ tends to zero, the limiting form for (5.5) is

$$(5.10)\qquad \begin{aligned} &\int_{0}^{\pi/2} \frac{\Omega}{\sin \phi}\, d\phi \\ &= 2\pi \sum_{k=0}^{\infty} \left[a_{2k+1} \cos (2k+1)\theta + b_{2k+1} \sin (2k+1)\theta\right] \sin (2k+1)\tau. \end{aligned}$$

Finally, applying to both members of (5.10) the operation

$$\frac{1}{\pi^2} \int_{\epsilon}^{\pi/2} \frac{d\tau}{\sin \tau}$$

and repeating the above reasoning yields the desired result:

$$(5.11)\qquad \Lambda[F_2] = \sum_{k=0}^{\infty} \left[a_{2k+1} \cos (2k+1)\theta + b_{2k+1} \sin (2k+1)\theta\right] = F_2(\theta). \text{ Q.E.D.}$$

Now, with the completion of the proof of Lemma 6, the asserted integral relation (1.2), or (2.10), has been verified. Furthermore, Lemma 6 can be expressed as a relation involving certain transforms. In particular, it is equivalent to

(5.12) LEMMA 7. *If* $F(\theta) \in H^{\alpha > 1/2}$, $F(\theta + \pi) = -F(\theta)$, *and*

$$(5.13)\qquad G(\tau) = \frac{1}{\pi} \int_{0}^{\pi/2} \frac{[F(\phi + \tau) - F(-\phi + \tau)]}{\sin \phi}\, d\phi,$$

then

$$(5.14)\qquad F(\phi) = -\frac{1}{\pi} \int_{0}^{\pi/2} \frac{[G(\tau + \phi) - G(\phi - \tau)]}{\sin \tau}\, d\tau.$$

6. **(2.4) solution of reduced wave equation.** The expression (2.4) can be written in the form

$$(6.1)\qquad \begin{aligned} u(x, y) &= \frac{1}{2\pi}\int_0^{2\pi} F(\phi)\cos(x\sin\phi - y\cos\phi)d\phi \\ &+ \frac{1}{2\pi^2}\int_0^{2\pi} d\phi F(\phi)\int_0^{\pi/2}\frac{\Gamma(x, y, \tau, \phi)}{\sin\tau}d\tau, \end{aligned}$$

where

$$(6.2)\qquad \begin{aligned} \Gamma(x, y, \tau, \phi) = \sin[x\sin(\tau+\phi) - y\cos(\tau+\phi)] \\ + \sin[x\sin(\tau-\phi) + y\cos(\tau-\phi)]. \end{aligned}$$

Certain differentiability properties of the function (6.1) will now be established. Because $\Gamma(x, y, 0, \phi)\equiv 0$,

$$(6.3)\qquad \begin{aligned} \int_0^X dx\int_0^{\pi/2}\frac{1}{\sin\tau}\frac{\partial\Gamma(x, y, \tau, \phi)}{\partial x}d\tau &= \int_0^{\pi/2}\frac{d\tau}{\sin\tau}\int_0^X\frac{\partial\Gamma(x, y, \tau, \theta)}{\partial x}dx \\ &= \int_0^{\pi/2}\frac{\Gamma(X, y, \tau, \phi)}{\sin\tau}d\tau - \int_0^{\pi/2}\frac{\Gamma(0, y, \tau, \phi)}{\sin\tau}d\tau. \end{aligned}$$

Consequently, as

$$\int_0^{\pi/2}\frac{1}{\sin\tau}\frac{\partial\Gamma(x, y, \tau, \phi)}{\partial x}d\tau$$

is a continuous function of (x, y), it follows from (6.3) that

$$(6.4)\qquad \frac{\partial}{\partial x}\int_0^{\pi/2}\frac{\Gamma(x, y, \tau, \phi)}{\sin\tau}d\tau = \int_0^{\pi/2}\frac{1}{\sin\tau}\frac{\partial\Gamma(x, y, \tau, \phi)}{\partial x}d\tau.$$

In a similar manner both the existence and the explicit integral form of any partial derivative of (6.1) can be established. Hence, since both $\cos(x\sin\phi - y\cos\phi)$ and $\Gamma(x, y, \tau, \phi)$ are solutions of (1.1), it follows that (6.1) is also a solution of the reduced wave equation.

REFERENCES

1. K. O. Friedrichs, *Seminar on recent developments in the theory of wave propagation*, New York University Notes, Sec. 3, 1949–1950, pp. 93–104.

2. F. Rellich, *Über das asymptotische Verhalten der Lösungen von $\Delta u+\lambda u=0$ in unendlichen Gebieten*, Jahresbericht der Deutschen Mathematiker Vereinigung vol. 53 (1943) pp. 57–65.

3. P. Schafheitlin, *Ueber die Darstellung der hypergeometrischen Reihe durch ein bestimmtes Integral*, Math. Ann. vol. 30 (1887) p. 159.

4. R. Courant and D. Hilbert, *Methods of mathematical physics*, vol. I, New York, Interscience, 1953, p. 475.

5. A. Zygmund, *Trigonometrical series*, Warsaw, 1935, p. 135.

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